第三次小考解答：

1.Determine the convergence of the series

$$\sum_{n=2}^{\infty}(-1)^n\frac{\ln n}{n-\ln n}$$

Sol:

this series is Alternating Series and satisfied：

1. $\frac{\ln n}{n-\ln n}$ is positive $\forall$n

2. $\mathrm{f(x)}=\frac{\ln x}{x-\ln x}, \mathrm{f}'(\mathrm{x})=\frac{1-\ln x}{(x-\ln x)^2}<0 \Rightarrow \mathrm{a_n}\geq \mathrm{a_{n+1}}$ when $\mathrm{n}>\mathrm{e}$

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3. $\lim_{n\to\infty}\frac{\ln n}{n-\ln n}=\frac{\frac{1}{n}}{1-\frac{1}{n}}=0$, by L'Hôpital

But：

$$\left|\sum_{n=2}^{\infty}(-1)^n\frac{\ln n}{n-\ln n}\right|=\sum_{n=2}^{\infty}\frac{\ln n}{n-\ln n}\geq\sum_{n=2}^{\infty}\frac{1}{n-\ln n}\geq\sum_{n=2}^{\infty}\frac{1}{n}$$

so, $\sum_{n=2}^{\infty}|a_n|$ is diverge.

That is,this series is converged conditionally.

2. Find the series' radius and interval of convergence. For what values of x does the series converge absolutely, or conditionally?

$$\text{(a)}\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x+2)^n}{n2^n}, \text{(b)}\ \sum_{n=2}^{\infty}\frac{x^n}{n\ln n}.$$

Sol:

(a) By ratio test：

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x+2)^{n+1}}{(n+1)2^{n+1}}\times\frac{n2^n}{(-1)^{n+1}(x+2)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x+2)}{2}\frac{n}{n+1}\right|<1\rightarrow -4<\mathrm{x}<0$$

by ratio test, where $-4<\mathrm{x}<0$,the series is converge absolutely.

Now, we check that when x=0 and x=-4.

where x=0：

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x+2)^n}{n2^n}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2)^n}{n2^n}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

this series is Alternating Series and satisfied：

1. $\sum_{n=1}^{\infty}(-1)^n a_n$, where a_n is positive.

2. $a_n > a_{n+1}$ $\left(\frac{1}{n} > \frac{1}{n+1}\right)$

3. $\lim_{n\to\infty} a_n = \frac{1}{n} = 0$

so, where x=0, the series converge conditionally.

where $x=-4$ ：

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x+2)^n}{n2^n}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-2)^n}{n2^n}=\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n}=\sum_{n=1}^{\infty}\frac{-1}{n}$$

So, where $x=-4$, the series is diverge.

In conclusion, Radius of convergence is 2,interval of convergence is$(-4,0]$

(b) By ratio test：

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)\ln(n+1)}\times\frac{n\ln n}{x^n}\right| = \lim_{n\to\infty}\left|x\frac{n}{n+1}\frac{\ln(n)}{\ln(n+1)}\right|$$

$$=\lim_{n\to\infty}|x|<1, -1<x<1.$$

其中, $\lim_{n\to\infty}\frac{\ln n}{\ln n+1} = 1$ by L'Hôpital **要寫，沒寫的人扣分。

by ratio test, where $-1<x<1$,the series is converge absolutely.

Now, we check that when $x=-1$ and $x=1$ converge or not.

where $x=-1$,

this series is Alternating Series and satisfied：

1. $\sum_{n=1}^{\infty}(-1)^n a_n$, where a_n is positive. $a_n = \frac{1}{n\ln n}$

2. $a_n > a_{n+1}$ $\left(\frac{1}{(n+1)\ln(n+1)} > \frac{1}{n\ln n}\right)$

3. $\lim_{n\to\infty} a_n = \frac{1}{n\ln n} = 0$

so, where $x=-1$, the series is converge conditionally.

where $x=1$,

by integral test,

$$\lim_{t\to\infty}\int_{2}^{\infty}\frac{1}{x\ln x}\mathrm{dx}=\lim_{t\to\infty}\int_{lnt}^{\infty}u\,\mathrm{du}=\infty \quad ** 要寫出來$$

so,where $x=1$,this series is diverge.

3.Find the Taylor series generated by $f=2^{x}$ at $x=1$.

Sol：

Taylor series of $f(x)=2^{x}$ at $x=1$：

$$f(x)=f(1)+\frac{f'(1)}{1!}(x-1)+\frac{f''(1)}{2!}(x-1)^2+\cdots$$

$$=2+\frac{2\ln 2}{1!}(x-1)+\frac{2(\ln 2)^2}{2!}(x-1)^2+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{2(\ln 2)^n}{n!}(x-1)^n$$

4.$f(x)=\begin{cases}x^2, & x<0\\ \sqrt{x}, & x\geq 0\end{cases}$,find f_x, f_y, f_{xy} and $f_{yx}$and state the domain for each partial derivative.

Sol：

At x=0,$f_x(0,y)=\lim_{h\to 0}\frac{f(0+h,y)-f(0,y)}{h}=\lim_{h\to 0}\frac{f(h,y)}{h}$ does not exist,because:

$$\lim_{h\to 0^-}\frac{f(h,y)}{h}=\lim_{h\to 0^-}\frac{h^2}{h}=0,\ \lim_{h\to 0^+}\frac{f(h,y)}{h}=\lim_{h\to 0^+}\frac{\sqrt{h}}{h}=\infty$$

$$f_x=\begin{cases}2x\ , & x<0\\ \frac{1}{2\sqrt{x}}, & x>0\end{cases}$$

$f_y=0\,,\forall y$

$f_{xy}=0$, for all points(x, y) such that $x\neq 0$.

$f_{yx}=0$,for all points(x,y).

5.Assuming that the equation $xe^{y}+ye^{z}+2\ln x-2-3\ln 2=0$ defined y as a differentiable function of x, find the value of dy/dx at the point (1,ln2,ln3)

Sol：

$$f_x(x,y,z)=e^{y}+\frac{2}{x}, f_y(x,y,z)=xe^{y}+e^{z}$$

$$\frac{\mathrm{dy}}{dx} = -\frac{\mathrm{f_x}}{\mathrm{f_y}} = -\frac{e^y + \frac{2}{x}}{xe^y + e^z}$$，帶入後得$$\frac{\mathrm{dy}}{dx}(1, \ln 2, \ln 3) = -\frac{4}{5}$$