2020/12/28 微積分小考(4) - §4.7~§5.6 滿分為 100 分

整體批改標準：符號標錯扣 2 分，說明不清楚、不完整扣 3 分，
過程沒寫或寫錯扣該題分數一半。

1.

(a) (10%)

① Let $n-1$ points $\{x_1, x_2, \dots, x_{n-1}\}$ between a and b and satisfying $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$ (2 分)

② A partition of $[a, b]$: $P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ (1 分)

③ The kth subinterval of P is $[x_{k-1}, x_k]$. (1 分)

④ The norm of a partition P, $\|P\|$ (2 分), the largest of all subinterval widths.

⑤ A Riemann sum for f on the interval $[a, b]$: $S_p = \sum_{k=1}^{n} f(c_k)\Delta x_k$ (3 分) for every $c_k \in [x_{k-1}, x_k], k = 1, \dots, n.$ (1 分)

(b) (10%)

Let $f(x)$ be a function defined on a closed interval $[a, b]$.
We say that a number I is the definite integral (2 分) of f over $[a, b]$ and that I is the limit (2 分) of the Riemann sums if the following condition is satisfied:

Given any number $\delta > 0$ there is a corresponding number such that for every partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ (2 分) and any choice of c_k in $[x_{k-1}, x_k]$,
we have $|\sum_{k=1}^{n} f(c_k)\Delta x_k - I| < \epsilon.$ (3 分)

2.

(a) (5%)

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$

(b) (15%)

PART1. (10 分)

If f is continuous on $[a,b]$ then

① $F(x)=\int_a^x f(t)dt$ is continuous on $[a,b]$ (3 分) and differentiable on (a,b), (3 分)

② and its derivative is $f(x)$; $F'(x)=\frac{d}{dx}\int_a^x f(t)dt=f(x)$. (4 分)

PART2. (5 分)

If f is continuous at every point of $[a,b]$ and F is any antiderivative of f on $[a,b]$, then $\int_a^b f(x)\,dx = F(x)|_a^b = F(b)-F(a)$.

3. (20%)

Let $\Delta x=\frac{b-0}{n}=\frac{b}{n}$ (3 分) and let

$x_0=0, x_1=\Delta x,\ x_2=2\Delta x,\dots,\ x_{n-1}=(n-1)\Delta x, x_n=n\Delta x=b.$

Let the c_k's be the right end-points of subintervals $\Rightarrow$ $c_1=x_1, c_2=x_2$, and so on. The rectangles defined have areas:

$$f(c_1)\Delta x=f(\Delta x)\Delta x=\left(\frac{\Delta x}{2}+1\right)(\Delta x)=\frac{1}{2}(\Delta x)^2+\Delta x$$

$$f(c_2)\Delta x=f(2\Delta x)\Delta x=\left(\frac{2\Delta x}{2}+1\right)(\Delta x)=\frac{2}{2}(\Delta x)^2+\Delta x$$

$$f(c_3)\Delta x=f(3\Delta x)\Delta x=\left(\frac{3\Delta x}{2}+1\right)(\Delta x)=\frac{3}{2}(\Delta x)^2+\Delta x$$

$$\vdots$$

$$f(c_n)\Delta x=f(n\Delta x)\Delta x=\left(\frac{n\Delta x}{2}+1\right)(\Delta x)=\frac{n}{2}(\Delta x)^2+\Delta x$$

Then

$$S_n=\sum_{k=1}^n f(c_k)\Delta x=\sum_{k=1}^n\left(\frac{1}{2}k(\Delta x)^2+\Delta x\right)$$

$$=\frac{1}{2}(\Delta x)^2\sum_{k=1}^n k+\Delta x\sum_{k=1}^n 1=\frac{1}{2}\left(\frac{b^2}{n^2}\right)\left(\frac{n(n+1)}{2}\right)+\left(\frac{b}{n}\right)n=\frac{1}{4}b^2\left(1+\frac{1}{n}\right)+b$$

$$\Rightarrow \int_0^b\left(\frac{x}{2}+1\right)dx=\lim_{n\to\infty}\left(\frac{1}{4}b^2\left(1+\frac{1}{n}\right)+b\right)=\frac{1}{4}b^2+b$$

未使用 limit of Riemann Sums 扣 10 分

4. (10%)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 x}}\left(\frac{d}{dx}(\sin x)\right) = \frac{1}{\sqrt{\cos^2 x}}(\cos x) = \frac{\cos x}{|\cos x|} = \frac{\cos x}{\cos x} = 1$$

5. (10%)

Let $u = x^3 + 1 \Rightarrow du = 3x^2 dx$ and $x^3 = u - 1.$

$$\int 3x^5\sqrt{x^3+1}\,dx = \int (u-1)\sqrt{u}\,du = \left(u^{3/2} - u^{1/2}\right)du$$

$$= \tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2} + C = \tfrac{2}{5}(x^3+1)^{5/2} - \tfrac{2}{3}(x^3+1)^{3/2} + C$$

6.

(a) (10%)

$y = 3 - x^2$ and $y = -1$

$\Rightarrow 3 - x^2 = -1 \Rightarrow x^2 = 4 \Rightarrow a = -2,\ b = -2$;

$f(x) - g(x) = (3 - x^2) - (-1) = 4 - x^2$

$$\Rightarrow A = \int_{-2}^{2}(4 - x^2)dx = \left[4x - \tfrac{1}{3}x^3\right]^2 = \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right) = \tfrac{32}{3}$$

(b) (10%)

Let $x = 0$ in $y = 3 - x^2 \Rightarrow y = 3$;

$f(y) - g(y) = \sqrt{3-y} - \left(-\sqrt{3-y}\right) = 2(3-y)^{1/2}$

$$\Rightarrow A = 2\int_{-1}^{3}(3-y)^{1/2}\,dy = -2\int_{-1}^{3}(3-y)^{1/2}(-1)\,dy$$

$$= (-2)\left[\frac{2(3-y)^{3/2}}{3}\right]_{-1}^{3} = \left(-\tfrac{4}{3}\right)\left[0 - (3+1)^{3/2}\right] = \left(\tfrac{4}{3}\right)(8) = \tfrac{32}{3}$$